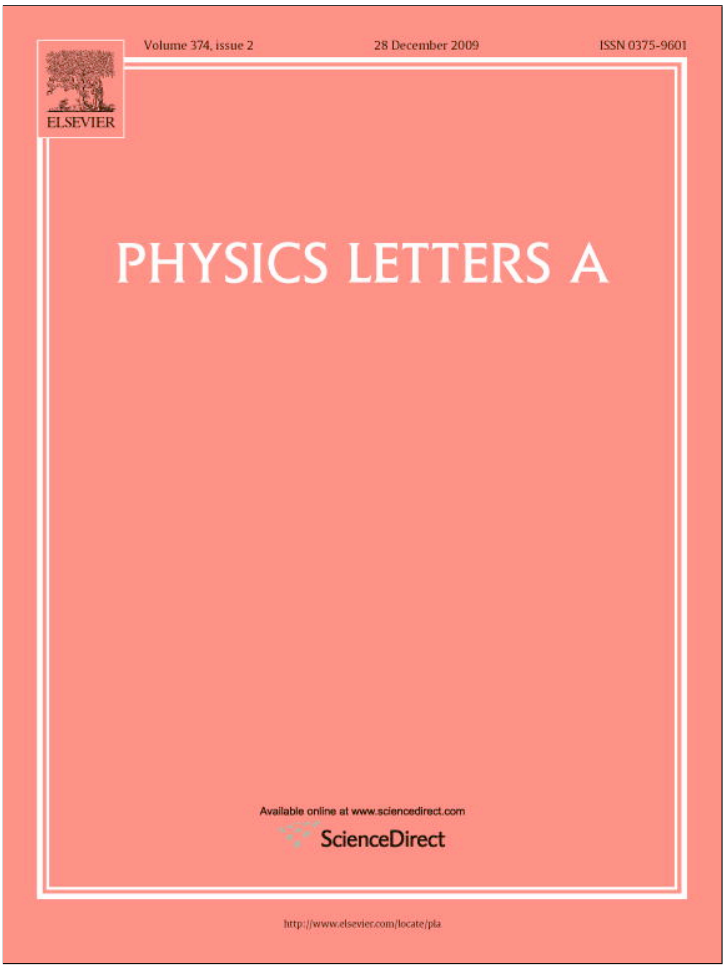

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On the one-dimensional Coulomb problem

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ABSTRACT

We analyse the one-dimensional Coulomb problem (1DCP) pointing out some mistaken beliefs on it. We show that no eigenstates of even or odd parity can represent states of the system. The 1DCP exhibits a sort of spontaneous breaking of parity. We also show that a superselection rule operates in the system. Such rule explains some of its peculiarities. We build the superpotential associated to the 1DCP.

The Hamiltonian ($q_e = \hbar = m = 1$)

$$H_{1DH} = -\frac{1}{2}\frac{d^2}{dx^2} - \frac{1}{|x|} \tag{1}$$

describes a one-dimensional quantum system with many uses despite its apparent simplicity [1–16]. The Hamiltonian (1) arises whenever one tries to describe systems with Coulomb-like interactions but constrained to move in one direction. For example, in describing atoms in superstrong magnetic fields [17], or excitonic motion in condensed matter [13,18], in studying above treshold ionisation of atoms on intense laser fields [19], or in describing electrons hovering above superfluids [18,20,21]; the 1DCP has many uses in atomic, molecular and condensed matter physics. Additionally, an essentially two-dimensional set of electrons trapped in the one-dimensional hydrogenic levels of (1) has been suggested as a possible realisation of a quantum computing device [22,23]. Behaviour under the $|x|^{-1}$ potential has been suggested as means for studying chaos in periodically driven atoms and for modelling excited states of hydrogen in a DC electric field. This is possible because the electronic distribution along the field [24,25] may be regarded [3,26–28] as interacting through (1). It is to be also noted that only references [15] and [16] deal with the scattering problem, all the others deal with the bounded part of the spectrum.

With the many uses of the one-dimensional Coulomb potential some claims made lately are troublesome. For example, it has been asserted that the potential of an 1DCP does not properly exist[1] or that, even if it exists, the Coulomb potential and its supersymmetric partner in one dimension are identical [3], or that there can be no superpotential associated with the $-1/|x|$ potential [29]. In this Letter, we show that such claims are not correct.

The peculiar features of the 1D Coulomb potential have prevented the calculation of critical electric moments in one dimension [30]. It has been suggested [1] that the eigenfunctions of the problem are only odd because, it is argued, the even eigenstates fail to be solutions of the problem. Contrary to these claims, the 1DCP admit no eigenfunctions symmetrical under inversion through the origin exhibiting a sort spontaneous breaking of parity [9,11]. That is, Nouri's conclusion [1] on the parity of the problem's eigenstates is untenable. Even the infinite energy ground state, shown not to exist a long time ago [8,9,31], is still in use [1,30]. Our results show that the conclusions on the superpotential in [3] and in [29] cannot be regarded as correct.

The Hamiltonian (1) is not in general self-adjoint. Self-adjoint four-parameter extensions have been derived in [32], such exten-

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[1] The mistaken claims began from the supposed existence of a strongly localised state of infinite binding energy as argued in [5].

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sion admits Hamiltonian (1) as one of its members [1,2,4,13,32]. H_D together with the matching condition $\phi(x)|_{x=0} = 0$ is self-adjoint. However, these considerations have had little effect on what people considers as properties of the system [29,30,33].

In this letter, we work not with the full self-adjoint extension of (1) but just with Hamiltonian (1) which, together with Dirichlet boundary conditions (i.e. $\phi(x)_{x=0} = 0$), coincides with the *self-adjoint* Hamiltonian H_D of [32]. The Hamiltonian (1) is precisely the one studied when analysing the 1D Coulomb problem [1–6,8–13,20,22,28,29,31,34,35]. Besides, the path integral approach, with a measure given by the kinetic energy, yields H_D as the only finite self-adjoint extension leading to a time evolution coincident with the evolution predicted by (1) [32]. Thence our choosing of (1) and not of any other among the four-parameter extension. Hamiltonian H_D is, besides, the only one among the elements of the self-adjoint extension which is defined over a set of functions making the kinetic energy operator to be also essentially self-adjoint [32]. The physical consequences of the mentioned properties of (1) single out this Hamiltonian, taken together with the Dirichlet boundary condition, as the self-adjoint Hamiltonian describing the one-dimensional Coulomb problem. These are the Hamiltonian and the boundary condition used in this work.

Let us pinpoint that the potential deserving the name one-dimensional Coulomb potential is not $-1/|x|$. In one dimension the potential must be the solution of the Poisson equation $\phi''_{1DC} = -4\pi\delta(x)$, where $\delta(x)$ is a Dirac delta function [36]. Solving that equation we find the 1D Coulomb potential as $\phi_{1DC}(x) = -2\pi|x|$. Therefore, $V_{1DC}(x) = 2\pi|x|$. However, this is not the potential usually referred to as the one-dimensional Coulomb potential; the potential to which such name is applied is $V_{1DH} = -1/|x|$, as in (1). This potential needs to be treated with care because of the singularity at $x = 0$ [2,22,32,37,38] but it definitively belongs, together with $\phi(x)|_{x=0} = 0$, in the self-adjoint Hamiltonian extending (1) [equations (2) and (3) in [32]]. The existence of a superpotential associated with $V_{1DH}(x)$ and hence of a supersymmetric extension of (1) can be exhibited explicitly, as we show in what follows.

The potential in Hamiltonian (1) [2,9,37] has been claimed to be its own supersymmetric partner [3], this claim is not correct as we show in this letter, see also [10,27]. It has been proved that the system violates the nondegeneracy theorem for one-dimensional problems [5,9,12]. A superselection rule has been shown to operate in the system [11,12,37,39]. The system hence illustrates superselection rules which may be of importance for quantum computing [22,40,41] and has also served for studying strong field effects in atoms [17,19,34,42]. Hamiltonian (1) has been also used for modelling electrons floating over liquid helium [18,20–22]. This use has been challenged and quantum defect theory has been proposed for better accounting of the experimental facts [13,43], but even the new proposal is closely related to (1).

In the case of the hydrogen atom within a magnetic, **B**, field, the motion of the electron can be regarded as a product of transverse Landau states times states depending on a coordinate parallel to **B** [17,44]. The motion at right angles to the magnetic field is confined to distances of the order of $\rho_c = \sqrt{c/B}$. As the intensity of **B** is increased, $\rho_c \to 0$ leaving only the motion along the magnetic field for a dynamical description [34]. For superstrong magnetic fields, the potential felt by the electron (assuming **B** pointing in the x-direction) can be approximated with (1). Any use of this model should take into account the barrier it poses to electrons in one dimension: they cannot move from $x < 0$ to $x > 0$ because of the mentioned superselection rule [11,27,37]. Some properties of the system can be explained by such superselection rule. For example, it prevents parity eigenstates in the system [11,12,27,37] thus confirming the result of [33] on the nonexistence of even parity states in the system but *disproving* the conclusion about the odd states. All parity eigenstates are forbidden in a system interacting through (1) [9,11,37,45]. The problem's Hamiltonian thus produces a toy version of the superselection rule explaining away the paradox of optical isomers of quantum chemistry [40,46,47]. For another example of a quantum system with a superselection rule and with possible importance for quantum computing see [48].

The eigenfunctions describing the bound states of the one-dimensional hydrogen atom [9,10,37] are

$$\phi_n^+(x) = F_n(x) \quad \text{and} \quad \phi_n^-(x) = F_n(-x), \tag{2}$$

where the function $F_n(x)$ is defined by

$$F_n(x) = \begin{cases} 2(-1)^{n-1}n^{-3/2}xL_{n-1}^1(2x/n)\exp(-x/n) & \text{if } x \geqslant 0, \\ 0 & \text{if } x < 0, \end{cases} \quad n = 1, 2, 3, \ldots \tag{3}$$

and the $L_m^1(x)$ are the generalised Laguerre polynomials used in [13]. Notice the vanishing of the eigenfunctions at $x = 0$ and the explicit separation between the $x > 0$ and the $x < 0$ regions. This feature, explicitly exhibiting a sort of spontaneous breaking of parity, is a manifestation of the superselection rule which prohibits any superpositions of the right ψ_n^+ with the left ψ_n^- eigenstates [9,11,48]. The operator generating the superselection rule is

$$\hat{S} = \sum_n \big(|n^+\rangle\langle n^+| - |n^-\rangle\langle n^-|\big), \tag{4}$$

where $|n^\pm\rangle$ correspond to the eigenfunctions of the system $\phi_n^\pm$, and n runs over all its admissible values. We pinpoint that the eigenfunctions (2) solve the problem with the matching condition $\phi_n^\pm(0) = 0$ and that they are different from others derived in the literature because most others are forced to be parity invariant. It may be useful to mention that the eigenfunctions given in equation (26) of [13] *are not* related to Hamiltonian (1) but just to some inequivalent isospectral Hamiltonian derived in that paper. So there is not necessarily any relationship with our eigenstates (2).

The energy eigenvalues of the problem are [9,13,16,45]

$$E_n = -\frac{1}{2n^2}, \quad n = 1, 2, 3, \ldots, \tag{5}$$

the ground state energy is thus $E_1 = -1/2$ — so the supposed ground state with infinite binding energy proposed in [5] does not exist. All the eigenfunctions of the problem vanish either to the right, the states ϕ_n^+, or to the left, the states ϕ_n^-, of the singularity at $x = 0$. If we define parity eigenstates

$$\phi_{\text{odd}} = (2)^{-1/2}(\phi_+ - \phi_-) \quad \text{and} \quad \phi_{\text{even}} = (2)^{-1/2}(\phi_+ + \phi_-), \tag{6}$$

these eigenstates would not be independent because their Wronskian determinant, $W(\phi_{\text{odd}}, \phi_{\text{even}})$, would then vanish. Therefore, parity eigenstates cannot exist for the self-adjoint extension H_D. Moreover, as can be established using a slight modification of the previous argument, any coherent superposition of the ϕ_n^+ with the ϕ_m^+ states is completely devoid of meaning [34,37,48]. This also establishes that a superselection rule [49] preventing superpositions between left and right eigenstates operates in the system. The existence of the superselection rule can be recasted as the impenetrability of the 1D Coulomb potential [12]. It is also in agreement with the result of [16] on the vanishing of the transmission amplitude for scattering off an 1DCP with no regard as to whether it is an attractive or a repulsive potential. Most previous papers on the 1DCP use different and incorrect eigenstates therefore some of their results are erroneous.

The eigenfunctions we have calculated allow us to recalculate the supersymmetric extension of the one-dimensional Coulomb

Hamiltonian. Such extension has an importance that cannot be underestimated since it may be relevant to the exact calculation of multisoliton solutions of certain high order partial differential equations [35,50–52]. It is also relevant in the search of isospectral potentials in quantum mechanics [3,13,51,52]. The supersymmetric partner potentials can be also profitable used for dealing with other physical systems as sometimes it is easier to solve the partner than the original potential [27]. For such reasons and given the many applications of the 1DH Hamiltonian, (1), the computation of the supersymmetric extension of the Hamiltonian of a particle interacting through a $-1/|x|$ potential is relevant.

The superpotential, $W(x)$, associated to the 1D Coulomb potential is given in term of the ground state $\phi_1(x)$, as [10,53]

$$W(x) = -\frac{\phi_1'(x)}{\phi_1(x)}, \tag{7}$$

taking both ground states, $\phi_1^+(x)$ for $x > 0$ and $\phi_1^-(x)$ for $x < 0$, we obtain

$$W(x) = \operatorname{sgn}(x) - \frac{1}{x}, \tag{8}$$

where $\operatorname{sgn}(x)$ is the signum function. Using $W(x)$, the corresponding partner potentials can be calculated [51] as $V_\pm(x) = (1/8)(dW/dx)^2 \pm (1/4)\,d^2W/dx^2$, that is,

$$V_+(x) = \frac{1}{2} - \frac{1}{|x|} + \frac{1}{x^2} \quad \text{and} \quad V_-(x) = -\frac{1}{|x|} + \frac{1}{2}, \tag{9}$$

where V_- is the one-dimensional Coulomb potential used in (1), V_{1DH}, but shifted so that its ground state energy is zero, and V_+ is the partner potential — in which a superselection rule also operates [27]. Equation (9) explicitly disproves the claim in Ref. [3] about the identity between V_+ and V_-.

The raising and lowering operators are

$$A^+ = -\frac{d}{dx} + W \quad \text{and} \quad A = \frac{d}{dx} - W, \tag{10}$$

where

$$\left[A, A^+\right] = 2\frac{dW}{dx} = 2\left(V^+ - V^-\right) = \frac{2}{x^2} \quad \text{and} \quad V_+ + V_- = 2W^2. \tag{11}$$

It is remarkable that eigenvalue problems for the Hamiltonians $H_\pm = -(1/2)\,d/dx^2 + V_\pm(x)$ for the partner potentials $V_-(x)$ and $V_+(x)$, would correspond to the defining equation for Whittaker functions [54]. It should be additionally shown that there exists a dense set of functions on which the above commutation relation is well defined (see, for example, [38]).

We have thus established that the one-dimensional potential V_{1DH} can be regarded as stemming from the superpotential $W(x)$ in Eq. (7) and that our results allow the calculation of the critical dipole moment in one dimension — in disagreement with what some people believes [30]. We have proved that both the claim of parity eigenstates, made in [1,5,29,33] and of the nonexistence of a superpotential for the 1DCP, discussed in [3], are mistaken. We have shown that the quantum 1DCP cannot admit even or odd eigenstates. This is due to the superselection rule operating in the system [11,37], causing also a sort of spontaneous breaking of parity. The scattering results in [16] help to corroborate such conclusion. Any conclusion arrived at by assuming the existence of parity invariant states, as in Refs. [1,3], cannot be correct.

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