Nonunitary representations of the SU(2) algebra in the Dirac equation with a Coulomb potential

R. P. Martínez-y-Romero and A. L. Salas-Brito
Laboratorio de Sistemas Dinámicos, Departamento de Ciencias Básicas, UAM-Azcapotzalco, Apartado Postal 21-726, Coyoacán 04000 DF, Mexico
Jaime Saldanña-Vega
Facultad de Ciencias, Universidad Nacional Autónoma de México, Apartado Postal 50-542, 04510 DF, Mexico
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A novel realization of the classical SU(2) algebra is introduced for the Dirac relativistic hydrogen atom defining a set of operators that allow the factorization of the problem. An extra phase is needed as a new variable in order to define the algebra. We take advantage of the operators to solve the Dirac equation using algebraic methods. A similar path to the one used in the angular momentum case is used; hence, the radial eigenfunctions so calculated comprise nonunitary representations of the algebra. One of the interesting properties of such nonunitary representations is that they are not labeled by integer nor by half-integer numbers, as occurs in the usual angular momentum representation. © 1999 American Institute of Physics.

I. INTRODUCTION

The unitary representations of groups and algebras are of a great interest in physics. We can mention the ubiquitous example of angular momentum or the SU(2) algebra. In this instance, as it is well known, the representations are labeled by two real numbers \( j \) and \( m \), which may take only integer or half-integer values; the representations depend only on two parameters, \( \theta \) and \( \phi \) say, which are defined in the compact sets \([0, \pi]\) and \([0, 2\pi]\), respectively. However, there are many other physically interesting groups or algebras that are not unitary. In such a case, the representations are not restricted to parameters defined in compact sets nor are its generators necessarily Hermitian, but they nevertheless can play an important role in physics. The Lorentz group being a very important example of the class of physically relevant noncompact groups whose algebra is not necessarily unitary.

It is our purpose in this article to introduce a realization of the cyclic SU(2) algebra in terms of non-Hermitian operators and then to use these operators to factorize and solve the relativistic Dirac hydrogen atom. The solution is obtained using algebraic methods, using the basic operators of the system following a route parallel to the unitary compact case, but introducing an extra variable (it is found below to play the role of a phase), which is required in our approach.

The use of algebraic techniques has been common for these sort of problems. For example, in the relativistic hydrogen atom a la Dirac, one approach has been the use of shift operators; whereas in the nonrelativistic case a successful approach requires the use of ladder operators and the factorization method. Our approach is more akin to the introduction of ladder operators than to the shift operator method customarily used for this problem.

The paper is organized as follows. In Sec. II we introduce the equations of the problem and define our notation. In Sec. III we construct the basic operators spanning the SU(2) algebra as a useful tool for the problem. In Sec. IV we define the inner product needed to investigate the...
II. THE DIRAC HYDROGEN ATOM

The symmetry algebra for the bound states of the nonrelativistic hydrogen atom, even in the classical case, is well known to be the SO(4), but in the relativistic case the situation is different, as it is well known. The algebra associated with the symmetry of the radial part of the problem can be regarded as SU(2), as occurs with the angular momentum. The beautiful thing that follows is that we can then proceed to solve the relativistic hydrogen atom following a method that essentially parallels the calculation of eigenfunctions and eigenvalues of the angular momentum at the only expense of introducing an extra phase.

Let us begin with the Dirac Hamiltonian of the hydrogen atom,

\[ H_D = \mathbf{\alpha} \cdot \mathbf{p} + \beta m - \frac{Ze^2}{r}, \]

where \( m \) is the mass of an electron and \( \mathbf{\alpha} \) and \( \beta \) are the standard Dirac matrices,

\[ \mathbf{\alpha} = \begin{pmatrix} 0 & \mathbf{\sigma} \\ \mathbf{\sigma} & 0 \end{pmatrix}, \quad \beta = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \]

where the 1’s and 0’s stand, respectively, for 2 \( \times \) 2 unit and zero matrices and the \( \mathbf{\sigma} \) is the standard vector composed by the three Pauli matrices \( \mathbf{\sigma} = (\sigma_x, \sigma_y, \sigma_z) \). Since the Hamiltonian (1) is invariant under rotations, the solutions of the problem can be written in the form

\[ \psi(r, \theta, \phi) = \frac{1}{r} \left( \begin{array}{c} F(r) \mathcal{Y}_{jm}(\theta, \phi) \\ iG(r) \mathcal{Y}^*_{jm}(\theta, \phi) \end{array} \right), \]

where \( \mathcal{Y}_{jm} \) and \( \mathcal{Y}^*_{jm} \) are spinor spherical harmonics of opposed parity. Parity is a good quantum number in the problem because the Coulomb potential is invariant under reflections; parity goes as \((-1)^l\) and, according to the triangle’s rule of addition of momenta, the orbital angular momentum \( l \) is given by \( l = j \pm \frac{1}{2} \). But, instead of working directly with parity, we prefer to introduce the quantum number \( \epsilon \), defined by

\[ \epsilon = \begin{cases} 1, & \text{if } l = j + \frac{1}{2}, \\ -1, & \text{if } l = j - \frac{1}{2}. \end{cases} \]

Thus \( l = j + \epsilon \frac{1}{2} \) in all cases; we also define \( l' = j - \epsilon \frac{1}{2} \). Accordingly, the spherical spinor \( \mathcal{Y}_{jm} \) depends on \( l \), whereas the spherical spinor \( \mathcal{Y}^*_{jm} \), which has an opposite parity, depends on \( l' \).

Writing the solutions in the form (3) completely solves the angular part of the problem.

Let us now address the radial part of the problem; we are interested in its bound states, then of the quantity \( k = \sqrt{m^2 - E^2} \) is positive definite; furthermore, let us define

\[ k = Ze^2, \quad \tau_j = \epsilon \left( j + \frac{1}{2} \right), \quad \nu = \sqrt{\frac{m - E}{m + E}}. \]
then, we can write the differential equations for the radial part of the problem, in terms of the dimensionless variable \( \rho = kr \), as

\[
\left( -\frac{d}{d\rho} + \frac{\tau_j}{\rho} \right) G(\rho) = \left( -\nu + \frac{\xi}{\rho} \right) F(\rho),
\]

(6)

and

\[
\left( +\frac{d}{d\rho} + \frac{\tau_j}{\rho} \right) F(\rho) = \left( \nu^{-1} + \frac{\xi}{\rho} \right) G(\rho);
\]

(7)

these equations are to be regarded as the initial formulation of our problem.

The first thing we want to do is to show that Eqs. (6) and (7) can be rewritten using a set of three operators whose commutation relations define a SU(2) algebra. To this end, let us first introduce the new variable \( x \) through the relation [but please notice that this change is not required for any of the conclusions that follow; see the second remark below after Eq. (15)]

\[
\rho = e^x,
\]

(8)

so \( x \) is defined in the open interval \((-\infty, \infty)\) and redefine the radial functions \( F(\rho) \) and \( G(\rho) \), introduced in Eqs. (6) and (7), in the form

\[
F(\rho(x)) = \sqrt{m + E} \left[ \psi_-(x) + \psi_+(x) \right],
\]

(9)

\[
G(\rho(x)) = \sqrt{m - E} \left[ \psi_-(x) - \psi_+(x) \right].
\]

(10)

In terms of the new functions, \( \psi_+(x) \) and \( \psi_-(x) \), we thus arrive at the following set of equations for our problem:

\[
\left[ \frac{d}{dx} + e^x - \frac{\xi E}{\sqrt{m^2 - E^2}} \right] \psi_+(x) = \left( \frac{\xi m}{\sqrt{m^2 - E^2}} - \tau_j \right) \psi_-(x),
\]

(11)

and

\[
-\left[ \frac{d}{dx} - e^x + \frac{\xi E}{\sqrt{m^2 - E^2}} \right] \psi_-(x) = \left( \frac{\xi m}{\sqrt{m^2 - E^2}} + \tau_j \right) \psi_+(x).
\]

(12)

This first-order system can be uncoupled, multiplying by the left the first equation [Eq. (11)] times the operators that appear between square brackets in the second equation and, \textit{vice versa}, by multiplying the second equation [Eq. (12)] times the operators that appear (between square brackets) in the first one. This procedure gives us the second-order system,

\[
\left[ \frac{d^2}{dx^2} + 2\mu e^x - e^{2x} - \frac{1}{4} \right] \psi_+(x) = \left( \tau_j^2 - \frac{\xi^2}{4} - \frac{1}{4} \right) \psi_+(x),
\]

(13)

and

\[
\left[ \frac{d^2}{dx^2} + 2(\mu - 1)e^x - e^{2x} - \frac{1}{4} \right] \psi_-(x) = \left( \tau_j^2 - \frac{\xi^2}{4} - \frac{1}{4} \right) \psi_-(x),
\]

(14)

where we have defined

\[
\mu = \frac{\xi E}{\sqrt{m^2 - E^2}} + 1.
\]

(15)
At this point there are several remarks that need be made. First, in the next section the seemingly odd term $-\frac{1}{2}$ in Eqs. (13) and (14) is shown to be necessary to construct the algebra. Second, as we said before, the change of variable from $\rho$ to $x$ is not really necessary for any of the calculations in the article, however, we prefer to work in the $x$ rather than in the $\rho$ variable because this choice simplifies the appearance of some of the equations and, mainly, because it makes the inner product introduced in Sec. III [in Eq. (33)] to look more familiar; but to make contact with the usual description, we sometimes, at our convenience, revert to the variable $\rho$. As a third remark, notice that we can regard Eqs. (13) and (14) as two eigenvalue equations, where the common eigenvalue $\omega$ is given by

$$\omega = \tau_j^2 - \zeta^2 - \frac{1}{2} = j(j + 1) - \zeta^2;$$

(16)

as it is obvious, we do not need to calculate $\omega$ because it follows directly from the radial symmetry of the problem and from the intensity of the interaction that is needed to set the scale. The fourth remark we want to make is that, as the minimum value of $j$ is $\frac{1}{2}$, then $\omega \geq 0$ for at least $Z = 1, 2, \ldots$, up to 118; for a discussion of the significance of this number see Ref. 4, p. 236.

III. AN OPERATOR ALGEBRA FOR THE DIRAC HYDROGEN ATOM

Our main purpose of this article is the construction of nonunitary representations of the SU(2) algebra for the Dirac hydrogen atom; let us introduce the operator

$$\Omega_3 = -i \frac{\partial}{\partial \xi},$$

(17)

depending exclusively on the new variable $\xi$, which is essentially an extra phase, as must be clear in what follows, and

$$\Omega_{\pm} = e^{\mp i \xi} \left( \frac{\partial}{\partial x} \pm i \frac{\partial}{\partial \xi} + \frac{1}{2} \right),$$

(18)

which depend both on $\xi$ and on the transformed “radial” variable $x$. These three operators satisfy the following algebraic relations:

$$[\Omega_3, \Omega_{\pm}] = \pm \Omega_{\pm},$$

(19)

and

$$[\Omega_+ , \Omega_-] = 2 \Omega_3.$$  

(20)

We can alternatively define the two operators, $\Omega_1$ and $\Omega_2$, as

$$\Omega_1 = \frac{1}{2}(\Omega_+ + \Omega_-), \quad \Omega_2 = \frac{1}{2j}(\Omega_+ - \Omega_-),$$

(21)

in terms of which the algebraic properties (19) and (20) read as

$$[\Omega_i, \Omega_j] = i \epsilon_{ijk} \Omega_k, \quad i, j, k = 1, 2, 3.$$  

(22)

It is now clear that the commutation relations (19) and (20)—or just (22)—correspond to an SU(2) algebra. To complete the discussion, we also need to introduce the Casimir operator of the algebra; let us consider the operator

$$\Omega^2 = \Omega_1^2 + \Omega_2^2 + \Omega_3^2;$$

(23)

we can easily show that (23) is indeed a Casimir for the algebra (19) and (20) [or (22)],
Here, as in the angular momentum case, we can regard $\Omega^2$ as the square of $\Omega = \Omega_1 \hat{1} + \Omega_2 \hat{j} + \Omega_3 \hat{k}$. To obtain the explicit expression of $\Omega^2$, it is better to calculate first the product $\Omega_+ \Omega_+$:

$$\Omega_+ \Omega_+ = \frac{\partial^2}{\partial x^2} e^{2x} - 2i e^x \frac{\partial}{\partial \xi} + i \frac{\partial}{\partial \xi} + \frac{\partial^2}{\partial \xi^2} - \frac{1}{4},$$

and obtain the Casimir operator from the relationship $\Omega^2 = \Omega_+ \Omega_+ + \Omega_3 (\Omega_3 - 1)$. We remark that we do not have a linear term in $\partial / \partial x$ in Eq. (22) because we choose the last constant in Eq. (18) precisely as $\frac{1}{2}$. We then easily conclude that the Casimir operator is given explicitly by

$$\Omega^2 = \frac{\partial^2}{\partial x^2} e^{2x} - 2i e^x \frac{\partial}{\partial \xi} - \frac{1}{4}.$$

Although we are not restricted to a compact set of parameters anymore due to the presence of the variable $x$, in analogy with the conventions used for the spherical harmonics $Y_\ell^m(\theta, \phi)$, we label the simultaneous eigenfunctions of $\Omega^2$ and $\Omega_3$ as $V_{\mu}(x, \xi)$, where the numbers $\omega$ are the eigenvalues of $\Omega^2$ and the numbers $\mu$ are the eigenvalues of $\Omega_3$. Therefore, we write

$$\Omega_3 V_{\mu}(x, \xi) = \mu V_{\mu}(x, \xi)$$

and

$$\Omega^2 V_{\mu}(x, \xi) = \omega V_{\mu}(x, \xi);$$

we thus have

$$V_{\mu}(x, \xi) = e^{i \xi \mu} P_{\mu}(x),$$

where, again, we have used the angular momentum analogy to write the $x$ functions as $P_{\mu}(x)$, resembling the usual notation for the Legendre polynomials $P_{\mu}(\theta)$. In this equation it becomes clear the role of $\xi$ as just an extra phase.

A comparison with Eqs. (13) and (14) tells us that

$$\psi_+(x) = P_{\mu}(x),$$

and

$$\psi_-(x) = P_{\mu - 1}(x).$$

The operators $\Omega_+$ are thus ladder operators for the problem; they move along the set of eigenfunctions, changing the eigenvalue $\mu$ to the eigenvalue $\mu \pm 1$, in a completely analogous way to the case of the angular momentum algebra,

$$\Omega_+ V_{\mu}(x, \xi) = V_{\mu \pm 1}(x, \xi);$$

the required proportionality constants are evaluated in Sec. V [Eq. (52)].

**IV. THE INNER PRODUCT AND RELATED PROPERTIES OF THE OPERATORS**

To establish the properties of the $\Omega$ operators and to construct the representations of the $SU(2)$ algebra they span, we need a properly defined inner product. Here we cannot longer rely on the angular momentum analogy, since two of the parameters of the algebra we purport to construct are defined over the noncompact interval ($-\infty, \infty$), making it completely different from the angular
momentum. This situation implies that not all the generators of the algebra are to be Hermitian (indeed, $\Omega_1$ is found to be Hermitian but $\Omega_1$ and $\Omega_2$ are found to be anti-Hermitian) and, as a consequence, that the corresponding group could not be unitary.

Let us consider the operation

$$
(\phi, \psi) = \int_{0}^{2\pi} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} \phi^*(\xi,x)\psi(\xi,x)dx.
$$

(33)

It is easy to show that Eq. (33) defines an inner product, since, as it is not difficult to prove, it satisfies the three basic properties: (i) If $\psi(x) = 0$, then $(\psi, \psi) = 0$; (ii) if $\psi(x) \neq 0$, then $(\psi, \psi) \geq 0$; (iii) if $c$ is any complex number, then $(\psi, c\phi) = c(\psi, \phi)$ and $(c\psi, \phi) = c^*(\psi, \phi)$.

To study the behavior of the generators of the algebra in terms of the inner product (33), we consider a certain function $\psi(x,\xi)$ associated with a certain fixed value $\mu_0$. We define the elements of the associated Hilbert space $H_{\mu_0}$ with functions of the form

$$
\psi(x,\xi) = e^{i(\mu_0 + m - n)\xi}\mathcal{F}(x),
$$

(34)

where $m$ and $n$ are integer numbers and $\mathcal{F}(x)$ is a well-behaved function depending only on $x$. As we exhibit in Sec. V, this is actually the general form of the functions inhabiting the Hilbert space of our problem.

With the help of the inner product (33), we can establish the properties of the $\Omega_a$ ($a = 1,2,3$) operators. Let us consider first $\Omega_3$; in this case the important product is

$$
(\psi^*, \Omega_3 \psi) = \delta_{m'-n',m-n}(\mu_0 + m - n) \int_{-\infty}^{\infty} \mathcal{F}(x)\mathcal{F}(x)dx;
$$

(35)

this last equation, due to the presence of the Kronecker delta, can be written as

$$
\int_{0}^{2\pi} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} \left[ i \frac{\partial}{\partial \xi} e^{-(\mu_0 + m' - n')\xi} \right] e^{-i(\mu_0 + m - n)\xi} \mathcal{F}(x)\mathcal{F}(x)dx,
$$

(36)

and this is precisely $(\Omega_3 \psi^*, \psi)$. We have in this way proved that $\Omega_3$ is a Hermitian operator:

$$
\Omega_3^* = \Omega_3.
$$

(37)

To study the operators $\Omega_{\pm}$, let us first consider $\Omega_+$ and evaluate the product

$$
(\psi^*, \Omega_+ \psi) = \int_{0}^{2\pi} \frac{d\xi}{2\pi} \int_{-\infty}^{\infty} e^{-i(\mu_0 + m' - n')\xi} \mathcal{F}(x)dx e^{i\left[ i \frac{\partial}{\partial x} e^{x} + i \frac{1}{2}\right]} e^{i(\mu_0 + m - n)\xi} \mathcal{F}(x)dx.
$$

(38)

To analyze this integral, it is simpler to split it in two parts. Let us consider first the $e^{i\xi(-\partial/\partial \xi + 1/2)}$ part; its contribution to the inner product (38) is

$$
\left(\mu_0 + m - n + \frac{1}{2}\right) \delta_{m'-n',m-n+1} \int_{-\infty}^{\infty} \mathcal{F}(x)\mathcal{F}(x)dx;
$$

(39)

this expression can be written as

$$
- \int_{0}^{2\pi} \frac{d\xi}{2\pi} \left[ e^{-i\xi} \left( i \frac{\partial}{\partial x} + \frac{1}{2}\right) e^{i(\mu_0 + m' - n')\xi} \mathcal{F}(x) \right] e^{i(\mu_0 + m - n)\xi} \mathcal{F}(x)dx.
$$

(40)

We now take care of the contribution of the term $e^{i\xi(-\partial/\partial \xi - e^x)}$. After a partial integration this contribution becomes
\[ \delta_{m-n'-1,m-n} \int_{-\infty}^{\infty} \left[ -\frac{\partial}{\partial x} + e^x \right] F(x) \bigg| F(x) \, dx. \]  

(41)

Taking together the two previous results, it is easy to see that the operator complies with \( \Omega_+ = -\Omega_- \). A similar calculation establish the analogous property \( \Omega_+^a = -\Omega_-^a \). Therefore, we have established that

\[ \Omega_+^a = -\Omega_-^a. \]  

(42)

We have proved that not all the operators are Hermitian, thence arriving to the anticipated results: We showed that the operator \( \Omega_3 \) is indeed Hermitian; as a consequence, we expect the range of the only parameter it depends on, \( \xi \), to be compact; this is certainly the case since \( \xi \in [0,2\pi] \), a compact set. The other two operators \( \Omega_1 \) and \( \Omega_2 \) depend upon \( x \), a variable defined over the noncompact set \( (-\infty, \infty) \), but here such operators are anti-Hermitian,

\[ \Omega_a^1 = -\Omega_a^2, \quad a = 1,2, \]  

(43)

as it is easy to show from Eqs. (21) and (42).

**V. REPRESENTATIONS OF THE SU(2) ALGEBRA (RADIAL EIGENFUNCTIONS OF THE PROBLEM)**

With the inner product defined in the previous section, we are in the position of introducing a complete orthogonal basis of simultaneous eigenfunctions for \( \Omega^2 \) and \( \Omega_3 \), which, accordingly, must carry a representation of the algebra—and, besides, they solve our radial eigenvalue problem [Eqs. (6) and (7)]. We have decided to choose the commuting operators in a similar fashion to the standard SU(2) case. Let us define then

\[ V_{\mu}^{\mu}(x, \xi) = |\omega \mu \rangle, \]  

(44)

where the kets \( |\omega \mu \rangle \) are both assumed orthogonal and normalized respect the inner product (33),

\[ \langle \omega' \mu' | \omega \mu \rangle = \delta_{\omega, \omega'} \delta_{\mu, \mu'}. \]  

(45)

Now, as \( \Omega_1 \) and \( \Omega_2 \) are not Hermitian, the Casimir operator \( \Omega^2 \) defined in (23) is not necessarily positive definite, we can nevertheless introduce a positive definite operator as

\[ \Omega_+ \Omega_- = -\Omega_1^2 - \Omega_2^2 + \Omega_3^2 = 2\Omega_3^2 - \Omega^2; \]  

(46)

the positivity of this operator allows us to show that

\[ 2\mu^2 \geq \omega, \]  

(47)

that is, \( |\mu| \) is bounded by below. As a consequence, there must exist a minimum value for \( |\mu| \), let us say \( \lambda = |\mu|_{\text{min}} \). We also know the that the ket \( |\omega \lambda \rangle \) is annihilated by \( \Omega_- \) or, equivalently, that \( \Omega_+ \Omega_- |\omega \lambda \rangle = 0 \); so

\[ \omega - \lambda^2 - \lambda = 0 \quad \text{or} \quad \omega = \lambda (\lambda + 1). \]  

(48)

Given Eq. (47), let us introduce a slight change in notation, writing \( \lambda \) instead of \( \omega \) in the eigenfunctions \( \mathcal{P}_\mu^\mu(x) \) of Eq. (29), that is, \( \mathcal{P}_\mu^\mu(x) \) is to be replaced by \( \mathcal{P}_\lambda^\lambda(x) \). Furthermore, as \( \lambda \) has to be positive and since \( \omega = \frac{\tau_j^2 - \xi^2}{\xi} - \frac{1}{2} \), we find that the minimum \( |\mu| \) value is

\[ |\mu|_{\text{min}} = \lambda = s + \frac{1}{2}, \]  

(49)

where we have defined
\[ s = \sqrt{j(j+1)} = \sqrt{(j+\frac{1}{2})^2 - \frac{1}{4}}. \]  

(50)

Notice that \( s \) is a real quantity for \( Z = 1,2,3,...,137 \).

However, the most important conclusion we can draw from Eqs. (49) and (50), is that \( \lambda \) no longer has to be an integer or half-integer number, as necessarily occurs in the unitary compact angular momentum case. We may see that this result is a direct consequence of both \( \Omega_1 \) and \( \Omega_2 \) not being Hermitian operators. So the operators introduced for the SU(2) algebra associated with the Dirac hydrogen atom lead naturally to nonunitary representations labeled by real numbers, or, at least, not necessarily integer nor half-integer ones.

For constructing the functions comprising the representations, let us introduce the constants \( C^\pm_{\mu} \) as follows [compare with Eq. (32)]:

\[ \Omega_\pm |\omega\mu\rangle = C^\pm_{\mu} |\omega\mu \pm 1\rangle; \]  

(51)

these constants can be explicitly evaluated from \( \langle \omega\mu |\Omega_+ |\omega\mu\rangle = C^-_{\mu} C^+_{\mu-1} \) and from \( (C^-_{\mu-1})^* = -C^-_{\mu} \); if we further assume these constants to be real, we easily get

\[ C^\pm_{\mu} = \pm \sqrt{\mu(\mu \pm 1) - \lambda(\lambda - 1)}, \]  

(52)

a result with a slightly different form than in the analogous angular momentum case.

The ground state of the hydrogen atom can be obtained from the equation \( \Omega_- |\lambda\lambda\rangle = 0 \) for the positive set of eigenvalues. Such an equation becomes

\[ e^{-i\xi} \left( \frac{\partial}{\partial x} + x e^{x} - \lambda + \frac{1}{2} \right) e^{i\lambda \xi} \mathcal{T}_x^{\lambda}(x) = 0. \]  

(53)

whose solution is

\[ \mathcal{T}_x^{\lambda}(x) = d_{\lambda} e^{x^2} \exp(-e^x) = d_{\lambda} e^{(\lambda-1/2)x} \exp(-e^x), \]  

(54)

where

\[ d_{\lambda} = \frac{\gamma(\lambda - 1/2)}{\sqrt{\Gamma(2\lambda - 1)}}, \]  

(55)

is a normalization constant and \( \Gamma(y) \) stands for the Euler-gamma function. Since \( \lambda \) is the lowest eigenvalue, in this instance we should have

\[ \psi_+(x) = \mathcal{T}_x^{\lambda}(x), \]  

(56)

and

\[ \psi_-(x) = 0. \]  

(57)

In terms of the radial variable \( \rho \), we can see from Eqs. (9) and (10) that the ground state solutions are

\[ F(\rho) = \sqrt{\frac{(m+E)}{2m(\lambda - 1/2)}} \rho^e e^{-\rho}, \]  

(58)

and

\[ G(\rho) = -\sqrt{\frac{(m-E)}{2m(\lambda - 1/2)}} \rho^e e^{-\rho}, \]  

(59)
normalized in the sense

$$\int_{0}^{\infty} (|F(\rho)|^2 + |G(\rho)|^2) d\rho = 1. \quad (60)$$

We can also see that in the case of negative eigenvalues the solutions behave as $-e^{\rho\rho_0}$, giving divergent behavior as $\rho \to \infty$. This behavior makes the negative energy solutions not integrable; therefore, we have to discard them if we want to describe physically realizable states.\(^{2,3,10}\)

The excited states can be obtained, as we do in the Appendix, by applying successively $\Omega_+$ to the ground state $|\lambda\lambda\rangle$, as it is customarily done for the spherical harmonics. The result involves the functions $\mathcal{P}_n^\lambda(\chi)$ introduced by Eqs. (29) and (34). The $\mathcal{P}_n^\lambda(\chi)$ are polynomials multiplied by the weight factor $W(\rho) = \rho^{k-\frac{1}{2}} e^{-\rho}$. This weight factor assures that the behavior of the big and the small components of the spinor are regular both at the origin as well as $\rho \to \infty$. As an illustration of the solutions discussed here, we quote in the Appendix the first few cases of $\mathcal{P}_{n+p}^\lambda(\chi)$, for $p = 0, 1, 2, 3, 4, 5$; there we also plot these polynomial parts of the functions [i.e., we plot $\mathcal{P}_{n+p}^\lambda(\chi)$ without the weight factors] for the first five excited energy levels. Notice that the polynomial part in the eigenfunction $\mathcal{P}_{n+p}^\lambda(\chi)$ is always of order $p$.

The matrix representations of the $\Omega_a, a = 1, 2, 3$, for $\lambda = s + \frac{1}{2} + p$, $p = 0, 1, 2, 3, ...$, are constructed from Eqs. (27), (28), (51), and (52). In contrast with the standard $SU(2)$ case, here the representations are non-Hermitian and infinite dimensional for each $\lambda$, except for $\Omega_3$. The matrix elements of the operator $\Omega_3$, including the negative eigenvalues series, are given by

$$\langle \omega\mu | \Omega_3 | \omega\mu' \rangle = \mu \delta_{\mu\mu'}, \quad (61)$$

where $\mu = \pm (\lambda + p), p = 0, 1, 2, ...$; therefore, its trace vanishes and the determinant of any element of the corresponding group with the form $\exp(i\Omega_3)\xi$, is always 1.

For the other two operators the only nonvanishing matrix elements are

$$\langle \omega\mu | \Omega_1 | \omega\mu \pm 1 \rangle = \mp \frac{i}{2} \sqrt{\mu(\mu \pm 1) - \lambda(\lambda - 1)}, \quad (62)$$

$$\langle \omega\mu | \Omega_2 | \omega\mu \pm 1 \rangle = - \frac{i}{2} \sqrt{\mu(\mu \pm 1) - \lambda(\lambda - 1)}. \quad (63)$$

This means again that the trace of both $\Omega_1$ and $\Omega_2$ vanish. Notice also that the determinant of a group element generated by $\Omega_1$ or $\Omega_2$ is 1 only for a purely imaginary parameter.

VI. THE ENERGY SPECTRUM

We can now evaluate the bound energy spectrum for the problem. As we mentioned in Sec. V, the bound state energy spectrum comprises only the positive series eigenvalues.\(^{2,10}\) Let us first express the energy in terms of the eigenvalue $\mu$ from Eq. (15),

$$\mathcal{E} = m \left[ 1 + \frac{s^2}{(\mu - 1/2)^2} \right]^{-1/2}; \quad (64)$$

then, for the case of positive eigenvalues—the only ones with physically appropriate eigenfunctions—we have that $\mu = \lambda + p$, where $p$ is a non-negative integer $p = 0, 1, 2, ..., \text{or, equivalently, that } \mu - \frac{1}{2} = s + p$. This gives precisely the energy spectrum of the relativistic hydrogen atom. To rewrite our result in a more familiar form, we need only to define the principal quantum number $n$ and the auxiliary quantity $\epsilon_j$ as follows:

$$n = j + \frac{1}{2} + p, \quad (65)$$

$$\epsilon_j = n - s - p = j + \frac{1}{2} - s; \quad (66)$$
we then finally conclude that \( \mu - \frac{1}{2} = s + p = n - \epsilon_j \), which gives precisely the well-known energy spectrum.\(^{10}\)

VII. CONCLUDING REMARKS

In summary, we have constructed a SU(2) algebra for the hydrogen atom in the Dirac formulation, introducing the Hermitian operator \( \Omega_3 \) and the anti-Hermitian operators \( \Omega_1 \) and \( \Omega_2 \). The result of all of this is that the representations are labeled by numbers \( \lambda \) that are neither integers nor half-integers, as in the case of the more familiar unitary representations. Nevertheless, the algebra introduced predicts precisely the energy eigenvalues and eigenfunctions of the Dirac hydrogen atom. One of the most noteworthy features of the representations reported here is the mixing of a spinorial angular momentum character, implying an equally spaced spectrum, with the energy requirements of the problem—requiring a differently spaced spectrum; the interplay of these two spectral requirements is basically reflected in the fact that the eigenvalues associated with Eqs. (13) and (14) follows from both the generic radial symmetry and the specific features of the interaction in the Dirac equation (1). From Eq. (15), it also follows that in the limit of a vanishing interaction, i.e., \( \xi \to 0 \), our representations collapse and, in this special case, \( \mu = 1 \) always. Such behavior is precisely as expected because there is no longer any restriction over the eigenvalues and thus the spectrum becomes continuous, corresponding to a free Dirac particle. The operator algebra we introduced allows an essentially algebraic solution of the Dirac hydrogen atom, which may have various applications.\(^3,4,6\)

It is to be noted also the possible connections that our formulation may have with systems with hidden supersymmetric properties,\(^11–14\) as we will discuss in a forthcoming article. The energy spectrum of the problem has some peculiarities that also appear in the spectrum of a Dirac oscillator;\(^12,13,15\) in particular, the equally spaced energy solutions for \( \psi_+(x) \) and \( \psi_-(x) \) resemble, respectively, the behavior of the big and the small components of the aforementioned system. As a result of this resemblance, we are studying the hidden supersymmetric properties and the superconformal algebra, in the sense of Refs. 13 and 16, associated with this problem. The similitude of Eqs. (13) and (14) with the corresponding ones for a Morse oscillator should be also noticed.\(^6\)

It is worth pinpointing that we are forced to introduce the new variable \( \xi \) in order to define the algebra; in terms of the solutions of the Dirac equation, \( \xi \) just plays the role of a phase. To exemplify, when we perform a "rotation" using \( \Omega_3 \), the big component changes from \( F(x) \propto [\psi_-(x) + \psi_+(x)] \) to \( F(x) \propto [e^{i(\mu - 1)\xi} \psi_-(x) + e^{i\mu\xi} \psi_+(x)] = e^{i\mu\xi}[e^{-i\xi} \psi_-(x) + \psi_+(x)] \). The phase \( e^{i\mu\xi} \) does not play any observable role, but the term \( e^{-i\xi} \) changes the relative phase between the \( \psi_+(x) \) and \( \psi_-(x) \) components of the eigenfunction and, in consequence, changes the radial function \( F_\rho \) itself although the energy spectrum is still invariant under such a transformation; this is a consequence of the fact that \( |\omega_\mu\rangle \) and \( \Omega_3 |\omega_\mu\rangle \) both correspond to the same eigenvalue \( \mu \). In a way, this resembles what happens when there are superselection rules in a system.\(^17–19\)

To finalize, let us comment that it is not widely known that the radial eigenfunctions of the Dirac hydrogen atom can be expressed in terms of generalized associated Laguerre polynomials, as was realized by Davis a long time ago.\(^7,20\) These polynomials, which are a generalization to noninteger indices of the usual associated Laguerre polynomials, are defined as\(^7\)

\[
\mathcal{L}_n^\alpha(x) = \frac{\Gamma(\alpha + p + 1)}{n! \Gamma(\alpha + 1)} \mathcal{F}_1(-p; \alpha + 1; x),
\]

where \( \Gamma(x) \) is again the Euler gamma function, \( p \) is a positive integer, and \( \mathcal{F}_1(-p; \alpha + 1; x) \) stands for the confluent hypergeometric function—having one of their arguments negative, the hypergeometric function reduces to a polynomial.\(^21\) The polynomial representation of the radial eigenfunctions introduced here is related to that used by Davis in Eqs. (A8) and (A9) of the Appendix.
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APPENDIX: THE POLYNOMIAL PART OF THE RADIAL EIGENFUNCTIONS

Our purpose in this appendix is to illustrate the behavior of the first few polynomials associated with the radial eigenfunction of the hydrogen atom in the base $|\lambda \mu\rangle$ and to relate this description to the old but little known results of Davis.\textsuperscript{7,11} We explicitly calculate the first six functions of the positive eigenvalue series; such functions are always of the form of the weight factor $W^{(\nu)}(\lambda \mu)$. The polynomials are plotted in Fig. 1. The radial eigenfunctions are essentially generated from the basic relationship $\Omega_{+}(\omega \mu) = C_{\mu}^+ |\omega \mu + 1\rangle$. In the $\rho$ variable, the first equation in the series can be written as

$$e^{i \xi} \left( -\frac{\partial}{\partial \rho} - \rho - i \frac{\partial}{\partial \xi} - \frac{1}{2} \right) e^{i \mu \xi} \rho^{\lambda} \rho(\rho) = C_{\mu}^+ e^{i \mu + 1} \xi \rho^{\lambda + 1} \rho(\rho),$$

(A1)

which is just the first term in the whole ascending series $\rho^{\lambda + p}(\rho) = \Omega_p \rho^{\lambda} \rho(\rho)$ used to recursively calculate (A2)–(A7). Note that the polynomial part of the function $\rho^{\lambda + p}(\lambda)$ is always of the form $\sum_{i=0}^{p} C_i(\lambda) \rho^i$, where $C_i(\lambda)$ is also an order $(p-i)$ polynomial in $\lambda$.

Starting with $\mu=\lambda$, the first few functions in the positive series are then given by

$$\rho^{\lambda} \rho(\rho) = W(\rho),$$

(A2)
\[
T_{\lambda}^{s^2+\frac{1}{2}+n'}(\rho) = \sqrt{\frac{2}{\lambda}}(\lambda - \rho)W(\rho), \tag{A3}
\]

\[
T_{\lambda}^{s^2}(\rho) = \frac{W(\rho)}{\sqrt{2\lambda(\lambda + 1/2)}}[2\rho^2 - (2\lambda + 1)(2\rho - \lambda)], \tag{A4}
\]

\[
T_{\lambda}^{s^2+\frac{1}{2}+n'}(\rho) = \frac{W(\rho)}{\sqrt{3\lambda(\lambda + 1/2)(\lambda + 1)}}[-2\rho^3 + 6\rho^2(\lambda + 1) - 3\rho(2\lambda^2 + 3\lambda + 1) + \lambda(2\lambda^2 + 3\lambda + 1)], \tag{A5}
\]

\[
T_{\lambda}^{s^2+\frac{3}{2}+n'}(\rho) = \sqrt{\frac{2}{3}}\frac{W(\rho)}{\sqrt{\lambda(\lambda + 1/2)(\lambda + 1)(\lambda + 3/2)}}\rho^4 - 2(3 + 2\lambda)\rho^3 + 3(3 + 5\lambda + 2\lambda^2)\rho^2
\]
\[-(3 + 11\lambda + 12\lambda^2 + 4\lambda^3)\rho + \left(\lambda^3 + 3\lambda^2 + \frac{11}{4}\lambda + \frac{3}{4}\lambda\right), \tag{A6}
\]

\[
T_{\lambda}^{s^2+\frac{5}{2}+n'}(\rho) = \frac{W(\rho)}{\sqrt{15\lambda(\lambda + 1/2)(\lambda + 1)(\lambda + 3/2)(\lambda + 5/2)}}[-\rho^5 + \left(5\lambda + \frac{29}{2}\right)\rho^4 - (10\lambda^2 + 51\lambda + 54)\rho^3
\]+ \left(10\lambda^3 + 66\lambda^2 + \frac{235}{2}\lambda + \frac{123}{2}\right)\rho^2
\]- \left(5\lambda^4 + 37\lambda^3 + \frac{319}{4}\lambda^2 + \frac{257}{4}\lambda + \frac{33}{4}\right)\rho + \lambda^5
\]+ \frac{15}{2}\lambda^4 + \frac{65}{4}\lambda^3 + \frac{105}{8}\lambda^2 + \frac{27}{8}\lambda \tag{A7}.}

The relationship of these polynomials to those used by Davis can be seen from Eqs. (13) and (14), putting \(\rho = \exp x\) and introducing the function \(v(\rho)\) according to \(\psi_+ = \rho^s \exp(\rho)\psi(\rho)\), and, finally, using \(\mathcal{L}(\rho) = \partial(\rho/\partial x)\), to get

\[
\rho \frac{d^2 \mathcal{L}}{d\rho^2} + \left[(2s + 1) - \rho\right] \frac{d\mathcal{L}}{d\rho} + \left[\frac{(s^2 + \xi^2 - \xi^2)}{\rho} + (\mu - s - 1/2)\right] \mathcal{L} = 0. \tag{A8}
\]

where \(\mu = s + \frac{1}{2} + n'\). This equation can be regarded as a generalization to a noninteger index of the usual associated Laguerre differential equation. The equation corresponding to \(\psi_-\) can be obtained in an analogous fashion. This equation reduces to the one used by Davis if and only if \(s^2 - \xi^2 - \xi^2\), a result that just recovers the definition in Eq. (50). Now we can give the explicit relationship between our representation of the radial eigenfunctions to that used by Davis as

\[
\psi_+(\rho) = T_{s^2+\frac{1}{2}+n'}(\rho) = \rho^s \exp(-\rho)\mathcal{L}^{s^2}_{n'+1}(2\rho),
\]

\[
\psi_-(\rho) = T_{s^2+\frac{3}{2}+n'}(\rho) = \rho^s \exp(-\rho)\mathcal{L}^{s^2}_{n'+1}(2\rho) \tag{A9}
\]

save for normalization factors (unimportant for the point at hand), where \(s = \lambda - \frac{1}{2}\), and where the generalized associated Laguerre polynomials used by Davis are defined in Eq. (67).