An operator solution for the hydrogen atom using the phase as an additional variable

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We discuss an operator solution for the bound states of the non-relativistic hydrogen atom. The method adds the phase of a state and its associated operator to the set of variables of the system. The augmented set of operators is found to form a closed set of commutation relations thus comprising an operator Lie algebra. From these relations, the energy spectrum and bounded radial eigenfunctions are calculated. Our approach is analogous to the one employed to compute the angular momentum spectrum and eigenfunctions but with operators satisfying an su(1,1) Lie algebra instead of su(2). This method, with the same operator algebra and minor modifications, may be used to solve the Dirac relativistic hydrogen atom. © 2007 American Association of Physics Teachers.

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I. INTRODUCTION

Algebraic methods are widely used for solving quantum problems. They have the distinct advantage of producing compact results as is apparent when the solution of the harmonic oscillator using raising and lowering operators is compared to the series solution of its Schrödinger equation. In addition, operator methods serve to introduce techniques that are important in advanced quantum theory and in other fields. In the dawn of quantum mechanics even the hydrogen atom was solved using an operator algebra. Nevertheless, the series expansion seems to be the preferred method of solution of the hydrogen atom in introductory and intermediate courses. The reason for this choice is that it can be applied to almost any central problem whereas algebraic methods are more problem specific. Nonetheless, a variety of alternatives for the solution of the hydrogen problem has been proposed.

In this paper we discuss an operator method for finding the bound states of the non-relativistic hydrogen atom; the method can also be applied with few changes to find the bound states of the relativistic hydrogen atom. To take advantage of the invariance under global changes of the phase of quantum states, we introduce the phase as an additional dynamical variable and a corresponding phase operator \( \Omega_0 \) in the problem. We further define two operators, \( \Omega_+ \) and \( \Omega_- \), which play the role of ladder operators. Guided by an analogy with angular momentum, we define two additional operators, \( \Omega_1 \) and \( \Omega_2 \), and cast the radial Hamiltonian of the hydrogen atom in the form of an eigenvalue problem for an operator \( \Omega_0 \), whose eigenvalues are the same as the eigenvalues of the angular momentum \( L^2/\hbar^2 \), namely, \( \ell (\ell + 1) \). The operator \( \Omega_0 \) commutes with all the \( \Omega_0(i=1,2,3) \) operators because the operator \( L^2 \) commutes with the angular momentum operators \( L_x, L_y, \) and \( L_z \). The commutation relations between all these operators comprise a closed operator algebra, the su(1,1) Lie algebra. If we fix the quantum number \( \ell \) and apply the ladder operator to the ground state of the algebra (there is one ground state for each \( \ell \)), we obtain both the radial eigenfunctions of the system and its energy spectrum.

Our method is different from previous proposals because it uses the phase of the eigenfunctions as an additional dynamical variable to which, as it is required of any physical variable, we associate an Hermitian operator \( \Omega_0 \). This association is physically relevant because we are taking advantage of a basic feature of any quantum state, its phase, as an ingredient in the solution. Our proposed operators involve only first-order derivatives and the solution is straightforward—we need to solve one differential equation [Eq. (48)]. The operator algebra we use should be not unfamiliar to students given its similarity with the angular momentum one. It is, furthermore, the well-known su(1,1) Lie algebra much employed in quantum optics. An additional bonus is that the method gives students practice with operator manipulations. The method can be applied to other problems including some relativistic systems. Thus, the approach discussed in this paper can become a valuable addition to the toolbox of methods for solving the hydrogen problem in intermediate or advanced courses.

II. THE RADIAL HAMILTONIAN

The radial Hamiltonian for the hydrogen atom is

\[
-\frac{\hbar^2}{2m} \frac{1}{r^2} \frac{d}{dr} \left( r^2 \frac{d}{dr} \right) - \frac{e^2}{r} + \frac{\hbar^2}{2m} \ell (\ell + 1)
\]

\[ R(r) = \frac{E}{r} \tag{1} \]

where \( \ell (=0,1,2,3,\ldots) \) is the orbital angular momentum quantum number, \( e \) is the electronic charge, \( m \) is the reduced mass, \( \hbar \) is Planck’s constant divided by \( 2 \pi \), and \( R(r) \) is the radial part of the eigenfunction with energy \( E < 0 \). We further define
the dimensionless quantity
\[ \rho = \frac{k}{2r}, \]
and introduce
\[ \lambda = \frac{2m\epsilon^2}{\hbar^2 k}. \]  

The radial equation (1) can be rewritten as
\[ \left[ \frac{1}{\rho^2} \frac{d}{d\rho} \left( \rho^2 \frac{d}{d\rho} \right) + \frac{2\lambda}{\rho} - \frac{\ell(\ell + 1)}{\rho^2} - 1 \right] R(\rho) = 0. \]  

We transform the radial eigenfunction by writing
\[ R(\rho) = F(\rho)/\rho^{1/2}, \]
and obtain
\[ \left[ \rho^2 \frac{d^2}{d\rho^2} + \rho \frac{d}{d\rho} + 2\lambda \rho - \rho^2 - \frac{1}{4} \right] F(\rho) = \ell(\ell + 1)F(\rho). \]  

Equation (7) is an eigenvalue equation with eigenvalues \( \ell(\ell + 1), \ell = 0, 1, 2, \ldots \), which are fixed by the angular invariance of the problem. Finally, we introduce \( x \) as
\[ x = \log \rho, \]
and rewrite Eq. (7) as
\[ \left[ \frac{d^2}{dx^2} + 2\lambda e^x - e^{2x} - \frac{1}{4} \right] F(x) = \ell(\ell + 1)F(x). \]  

In Eq. (9), as in the original Eq. (1), we can change the phase of \( F \) without affecting the physics described by it. This invariance under global a change of phase is an important feature of any quantum state.

### III. THE OPERATOR ALGEBRA

Let us focus on the phase \( \phi \in [0, 2\pi] \) of \( F \). We can associate \( \phi \) with the operator
\[ \Omega_\lambda = -i \frac{\partial}{\partial \xi}, \]
which comes immediately to mind if we desire an operator with eigenfunctions \( e^{i\phi} \) and eigenvalues \( \phi \). Regarding the phase as a variable is possible because a global phase change has no consequences. Thus, the eigenstates of the hydrogen atom should be eigenstates of the operator \( \Omega_\lambda \), the eigenvalue being the phase of the state. This operator is Hermitian, as we check in Sec. IV. We also introduce two more operators:
\[ \Omega_+ = i e^{+i\phi} \left( \frac{\partial}{\partial x} - e^i - \frac{i}{\partial \xi} + \frac{1}{2} \right) \]
and
\[ \Omega_- = i e^{-i\phi} \left( \frac{\partial}{\partial x} + e^i + \frac{i}{\partial \xi} + \frac{1}{2} \right), \]
depending both on \( \xi \) and on the transformed variable \( x \).

By using the definitions Eqs. (10)–(12), we can show that the \( \Omega \) operators satisfy the commutation relations
\[ [\Omega_\lambda, \Omega_\pm] = \pm \Omega_\lambda \]
and
\[ [\Omega_\pm, \Omega_\pm] = -2\Omega_\lambda. \]

The resemblance of Eqs. (13) and (14) to the commutation relations between the angular momentum operators \( L_\lambda \) and \( L_\pm \) is readily apparent, the only difference being the sign in Eq. (14). This kind of closed commutation relations between the operators defines a Lie algebra. For the familiar case of the angular momentum operators, the operator algebra is called an su(2) Lie algebra. In our case the commutation relations, Eqs. (13) and (14), define what is called an su(1,1) Lie algebra. All the \( \Omega \) operators are a function of two variables, \( x \) and \( \xi \), so our realization of su(1,1) is two-dimensional.

Inspired by the angular momentum analogy, we define two more Hermitian operators, \( \Omega_1 \) and \( \Omega_2 \), as
\[ \Omega_1 = \frac{1}{2} (\Omega_+ + \Omega_-), \quad \Omega_2 = \frac{1}{2i} (\Omega_+ - \Omega_-), \]
which, together with \( \Omega_3 \), satisfy
\[ [\Omega_1, \Omega_2] = -i\Omega_3, \quad [\Omega_2, \Omega_3] = i\Omega_1, \quad [\Omega_3, \Omega_1] = i\Omega_2. \]

Note the minus sign in the first commutator in Eq. (16); this sign is the only difference with the corresponding expressions of the angular momentum algebra. The commutation relations (16) are an alternative form of expressing the su(1,1) Lie algebra.

\( L^2 \) commutes with all the other operators and is the Casimir operator of the angular momentum algebra. In our case, guided by the similarity between the su(1,1) and the su(2) algebras, we can search for an equivalent operator. A little searching and hints from Eq. (16) show that the desired Casimir operator \( \Omega_c \) is
\[ \Omega_c = -\Omega_1^2 - \Omega_2^2 + \Omega_3^2 = \frac{\partial^2}{\partial x^2} - e^{2i} - 2i e^i \frac{\partial}{\partial \xi} - \frac{1}{4}, \]
which satisfies
\[ [\Omega_c, \Omega_i] = 0 \quad \text{for} \quad i = 1, 2, 3. \]

In our problem, as can be done with the operators \( L^2 \) and \( L \) in the angular momentum case, we can regard \( \Omega_c \) as the square of the vector operator \( i\hat{\Omega}_1 i + i\hat{\Omega}_2 j + \Omega_3 \hat{k} \), where \( i, j, \) and \( k \) are, respectively, the unit vectors in the \( x, y, \) and \( z \) directions.

### IV. EIGENVALUES AND EIGENFUNCTIONS OF \( \Omega_c \) AND \( \Omega_3 \)

The next step is to calculate the complete set of the eigenvalues and simultaneous eigenfunctions of \( \Omega_c \) and \( \Omega_3 \). To this end we write the eigenfunctions as \( V_\lambda(x, \xi) \), where \( x \) represents the transformed radial variable, \( \rho = e^x \), and \( \xi \) is the variable appearing in Eq. (10). The eigenvalues of \( \Omega_1 \) and \( \Omega_3 \) are, respectively, \( \lambda \) and \( \omega \); that is,
\[ \Omega_1 V_\lambda(x, \xi) = \lambda V_\lambda(x, \xi) \]
and
\[ \Omega_3 V_\lambda(x, \xi) = \pm \Omega_\lambda V_\lambda(x, \xi) \]
\[ \Omega_{\lambda} V_{\omega}^{\lambda}(x, \xi) = \omega V_{\omega}^{\lambda}(x, \xi). \]  

From the definition of \( \Omega_{3} \) as \(-i\partial/\partial \xi\), we easily find 
\[ V_{\omega}^{\lambda}(x, \xi) = e^{i\lambda \xi} F_{\omega}^{\lambda}(x) \] 
from Eq. (9). We thus conclude that the eigenfunctions for the Coulomb problem are \( F(x) = F_{\omega}^{\lambda}(x) \), so, according its definition [Eq. (6)] 
\[ R(x) = e^{-i\xi x} F_{\omega}^{\lambda}(x). \]  

In terms of \( \rho \), 
\[ R(\rho) = \frac{1}{\rho^{1/2}} F_{\omega}^{\lambda}(\rho). \]  

Hence, according to Eq. (7) or (9), the eigenvalues \( \omega \) must be 
\[ \omega = \ell (\ell + 1). \]  

So far we have recast the radial non-relativistic Schrödinger equation in the form of the Casimir operator of a su(1,1) algebra, with eigenvalues given by \( \omega = \ell (\ell + 1) \). The useful features following from these rather simple properties are described in the following.

Note the similarities between our equations and those corresponding to angular momentum.\(^2\)\(^4\) The mathematical structure of the solutions should reflect this resemblance. In particular, we expect that the \( F_{\omega}^{\lambda}(x) \) functions depend on the \( \omega \) and \( \lambda \) eigenvalues, as is the case with the spherical harmonics which depend on both the \( \ell \) and \( m \) quantum numbers.\(^18\)\(^20\) However, there are some differences. Note that in our case \( \xi \) is not a variable defined in a bounded and closed interval, that is, it is not a compact variable, as is the case for the angular variables used in the spherical harmonics, which can be restricted to take values in the compact interval between 0 and \( 2\pi \).\(^22\) This property is only shared by the phase variable \( \xi \) which also is a compact variable. The choice of the notation for the eigenvalue of \( \Omega_{3} \) as \( \lambda \) and \( F_{\omega}^{\lambda}(x) \)—same symbol as in Eq. (9)—is not an accident.

Despite the different sign in the operator algebra with respect to the angular momentum operator, \( \Omega_{\lambda} \) still plays the role of a ladder operator. This feature can be verified from Eq. (13), because 
\[ \Omega_{\lambda} V_{\omega}^{\lambda}(x, \xi) = (\Omega_{\lambda} \Omega_{3} + \Omega_{3} \Omega_{\lambda}) V_{\omega}^{\lambda}(x, \xi) = (\lambda \pm 1) \Omega_{\lambda} V_{\omega}^{\lambda}(x, \xi). \] 
As in the angular momentum case, we conclude that \( \Omega_{\lambda} \) changes the eigenvalue \( \lambda \) to \( \lambda \pm 1 \),
\[ \Omega_{\lambda} V_{\omega}^{\lambda}(x, \xi) \propto V_{\omega}^{\lambda \pm 1}(x, \xi). \]  

To check the hermiticity and other properties of the \( \Omega \) operators, we use the product
\[ (\phi|\psi) = \int_{0}^{2\pi} \frac{d \xi}{2\pi} \int_{-\infty}^{\infty} \bar{\phi}^{*}(\xi, x) \psi(\xi, x) dx, \]  
where \( \phi(\xi, x) \) and \( \psi(\xi, x) \) are assumed to be periodic functions over the interval \( \xi \in [0, 2\pi] \) and are also assumed to vanish as \( x \to \pm \infty \). From Eq. (25) we find that \( \Omega_{3} \) is Hermitian,
\[ \Omega_{3}^{\dagger} = \Omega_{3}. \]  

The two operators \( \Omega_{1} \) and \( \Omega_{2} \) are a little different because they are not defined over a compact set of variables, but also depend on \( x \). From Eqs. (11), (12), and (15) we have
\[ \Omega_{1} = i \left( \cos \frac{\partial}{\partial x} - i \sin \frac{\partial}{\partial \xi} + \sin \frac{\partial}{\partial \xi} + \frac{1}{2} \cos \xi \right). \]  

Note that the term \( \sin \frac{\partial}{\partial \xi} + \frac{1}{2} \cos \xi \) just changes its sign under the adjoint operation because \( \sin \frac{\partial}{\partial \xi} + \frac{1}{2} \cos \xi = -\sin \frac{\partial}{\partial \xi} - \frac{1}{2} \cos \xi. \) We can also see that the other two terms \( \cos \xi, \frac{\partial}{\partial \xi} - i \sin \exp x \) behave in the same way. Thus,
\[ \Omega_{1}^{\dagger} = \Omega_{1}. \]  

A repetition of the previous analysis for \( \Omega_{2} \) shows that
\[ \Omega_{2}^{\dagger} = \Omega_{2}. \]  

That is, all \( \Omega_{i}, i = 1, 2, 3 \) operators are Hermitian.

Because \( \Omega_{1} \) and \( \Omega_{3} \) commute, we can obtain a complete orthogonal basis of their simultaneous eigenfunctions. We write the basis states as the kets
\[ V_{\omega}^{\lambda}(x, \xi) = |\omega \lambda \rangle. \]  

As usual, we assume this basis to be orthogonal and normalized,
\[ (\omega' \lambda' | \omega \lambda) = \delta_{\omega \omega'} \delta_{\lambda \lambda'}. \]  

In contrast with \( L_{2} \) the Casimir operator \( \Omega_{3} \) is not positive definite, as we can see from \( \Omega_{3}^{\dagger} = \Omega_{3}^{2} = \Omega_{3} \) which is obviously not positive definite. Nevertheless, the operator \( \Omega_{2}^{\dagger} = (\Omega_{2}^{2} + \Omega_{3}^{2} + \Omega_{3}^{2}) \) is positive. So, from the relation
\[ \Omega_{2}^{2} = \Omega_{3}^{2} + \Omega_{3}^{2} + \Omega_{3}^{2} = 2 \Omega_{3}^{2} - \Omega_{3}, \]  
we have
\[ 2 \lambda^{2} \geq \omega. \]  

The immediate consequence is that \( |\lambda| \) must have a minimum, say \( |\lambda|_{\text{min}} \). We have two choices, either \( \lambda > 0 \) or \( \lambda < 0 \). If we choose \( \lambda > 0 \), then \( \lambda \) has a minimum value. If we choose \( \lambda < 0 \), then \( \lambda \) has a maximum value. We first take \( \lambda > 0 \) and define \( N = \lambda_{\text{min}} \). Because \( \Omega_{\lambda} \) acts over the orthogonal basis \( |\omega \lambda \rangle \) as a ladder operator, we have that each time we use \( \Omega_{\lambda} \), we descend one step. Because \( N \) is the minimum value, \( \langle \omega \lambda | \omega N \rangle = 0 \) or, equivalently, \( \langle \omega \lambda | \omega N \rangle = 0 \). From the fact that \( \Omega_{3} \omega \lambda = \omega \lambda \pm 1 \), we obtain
\[ \Omega_{3} \omega \lambda = - \omega \lambda + \Omega_{3} - \Omega_{3}. \]  

Equivalently, we have that \(-\omega + N^{2} - N = 0\), or
\[ \omega = \ell (\ell + 1) = N(N-1). \]  

The solutions of Eq. (35) are \( N = -\ell \) and \( N = \ell + 1 \); the former is not allowed because we are assuming \( \lambda > 0 \) and because it can be shown to lead to solutions that diverge at the origin.\(^16\) Thus, the only admissible choice is
\[ N = \ell + 1. \]  

As \( \ell \) varies from 0 up to infinity in unit steps, we must have that \( N = 1, 2, 3, \ldots \); in other words, \( N \) plays the role of the principal quantum number of the hydrogen atom.\(^23\) We have thus found the eigenvalues of the operator algebra. We can relabel the eigenfunctions of the problem as
\[ |\omega \lambda \rangle \rightarrow |N \ell \rangle, \]  

and to label the eigenkets of the problem in a more standard way.

What happens when we consider the negative set of eigenvalues \( \lambda \)? In that case \( |\lambda|_{\text{min}} = M < 0 \) is the largest eigen-
value. Consequently, we must have that \( \Omega | \omega M \rangle = 0 \) for this state, or \( \Omega \Omega | \omega M \rangle = 0 \). The solution to this equation is \( M(M+1) = \ell(\ell + 1) \), which is impossible because \( M \) is less than zero. Thus, the solutions for \( M < 0 \) are not acceptable and must be discarded.

V. THE ENERGY SPECTRUM AND THE RADIAL EIGENFUNCTIONS OF THE HYDROGEN ATOM

What about the energy spectrum of the hydrogen atom? The energy eigenvalues follow immediately by setting \( \lambda = N \) and using Eqs. (2) and (4),

\[
E = - \frac{e^2}{2a_0N^2}, \quad N = 1, 2, 3, \ldots
\]

which is the Balmer formula, where \( a_0 = \hbar^2/m_e c^2 \) is the Bohr radius.

We note that the operator algebra we used to solve the non-relativistic hydrogen atom is the same as the algebra needed to solve the Dirac relativistic hydrogen atom.\textsuperscript{15,16,27}

To find the eigenfunctions we start with the ground state \( |N=1, \ell=0 \rangle = 0 \). From the equation \( \Omega = N=1, \ell=0 \rangle = 0 \) for the positive set of eigenvalues, we obtain

\[
e^{-i\xi} \left( \frac{\partial}{\partial x} + e^i + i \frac{\partial}{\partial \xi} + \frac{1}{2} \right) e^{i\xi} F_1^0(x) = 0,
\]

whose solution is

\[
F_1^0(x) = C_1^0 e^{x^2} \exp(-e^x) \quad \text{or} \quad F_1^0(\rho) = C_1^0 \rho^{1/2} e^{-\rho},
\]

where \( C_1^0 \) is a normalization constant. If we use the definition (2) \( k = 2/a_0 N \) and, thus, \( \rho = (k/2) r = (r/a_0 N) \). From Eq. (6) we obtain the ground state eigenfunction

\[
R_1^0(r) = C_1^0 e^{-r/a_0},
\]

which is the \( N=1, \ell=0 \) solution of the non-relativistic hydrogen atom. Remember that our definition of \( \rho \) involves an extra \( \frac{1}{2} \) compared to other authors (for example, Ref. 18).

Once we have the ground state for a given \( \ell \), we can obtain the entire set of states with \( \ell \) fixed for all \( N = \ell + 1, \ell + 2, \ldots \) from the equation \( \Omega | \ell, \ell \rangle = |N, \ell \rangle \). For instance, we can obtain the state \( |N=2, \ell=0 \rangle \). From Eq. (40) we have \( |0\rangle = e^{i\xi} e^{x^2} \exp(-e^x) \), except for a normalization constant. Thus, using \( \Omega_\ell \) in Eq. (11), we have

\[
\Omega_\ell |0\rangle = i e^{i\xi} \left( \frac{d}{dx} - e^x + 1 + \frac{1}{2} \right) e^{x^2} e^{-e^x}.
\]

After some minor manipulations, the result is

\[
|2, 0\rangle = C_2^0 e^{2i\xi} F_2^0(x) = C_2^0 e^{2i\xi} e^{x^2} e^{-e^x}(1 - e^x).
\]

For \( N=2 \) we have from the definitions (2)–(4) that \( \rho = (r/2a_0) \) which gives

\[
R_2^0(r) = C_2^0 \left( 2 - \frac{r}{a_0} \right) e^{-r/2a_0},
\]

which is the \( N=2, \ell=0 \) radial wave function of the non-relativistic hydrogen atom.\textsuperscript{18,20} The constant \( C_2^0 \) is a normalization constant.

We can continue in this way and obtain \( R_3^0(r) \). From

\[
|3, 0\rangle \sim \Omega_\ell |2, 0\rangle = i e^{i\xi} \left( \frac{d}{dx} - e^x + 1 + \frac{1}{2} \right) e^{x^2} e^{-e^x}(1 - e^x).
\]

If we introduce a normalization constant, we have

\[
|3, 0\rangle = C_3^0 e^{3i\xi} F_3^0(x) = C_3^0 \left[ 3 - 6 e^x + 2 e^{2x} \right] e^{3i\xi} e^{x^2} e^{-e^x}.
\]

From our definitions Eqs. (2)–(4), we find that \( \rho = (r/3a_0) \), or

\[
R_3^0(r) = C_3^0 \left[ 27 - 18 \frac{r}{a_0} + 2 \left( \frac{r}{a_0} \right)^2 \right] e^{-r/3a_0}.
\]

Again we obtain the correct wave function.\textsuperscript{18,20} In this way we can obtain all the \( s \) state wave functions.

How do we proceed with the rest of the eigenfunctions? We must obtain the base state of the algebra for \( \ell \) fixed. The base state of the algebra is the state that satisfies the condition \( \Omega |\lambda_{\min}, \ell \rangle = 0 \). This state is \( |N, N-1 \rangle \). Thus, the condition \( \Omega |N, N-1 \rangle = 0 \); that is, the operator \( \Omega_\ell \) annihilates such a state, and we have

\[
ie^{-i\xi} \left( \frac{\partial}{\partial x} + e^i + i \frac{\partial}{\partial \xi} + \frac{1}{2} \right) e^{iN\xi} F_{N-1}^0(x) = 0,
\]

whose solution is given by

\[
|N, N-1 \rangle \sim e^{iN\xi} e^{(N-1)/2} x^{-e^x}.
\]

Note that Eq. (48) is the only differential equation that we need to solve. The radial eigenfunctions follow from its solution.

From the definitions (2) and (38) we obtain for \( \rho = e^x \) the general result \( \rho = (2r/a_0 N) \). From the radial equation (6) we obtain

\[
R_a^{N-1}(r) = C_{N-1}^0 \left[ r^{-N-1} e^{-2a_0 N} \right],
\]

which is the correct result.\textsuperscript{18,20} From Eq. (50) we can obtain the radial wave functions for any value of \( \ell = 0, 1, 2, \ldots \) by applying successively the operator \( \Omega_\ell \) to the base state. For example, the state with \( N=2, \ell = 1 \) is

\[
|R_2^1(r) \sim C_2^1 e^{3/2} x^{-e^x}.
\]

or

\[
R_2^1(r) \sim C_2^1 e^{-r/2a_0}.
\]

We now discuss the normalization constants. We can always calculate the normalization constant by noticing that the solution can be cast in the form \( R_\ell^0(\rho) = C_\ell^0 \rho e^{-\rho L_{2i\xi}^{(1)}(2\rho)} \). But to be consistent, the method described here should provide a way to calculate the normalization constants. Because all the wave functions are obtained from the application of \( \Omega_\ell \) to the base state, that the solutions are always of the form \( \rho e^{-\rho P(2\rho)} \), where \( P(2\rho) \) is a polynomial in \( \rho \). (We know that this polynomial is a Laguerre polynomial.) Thus, to calculate the normalization constant, we need to evaluate

\[
\int_0^\infty R^2(2\rho) \rho^2 d\rho,
\]

which implies the calculation of quantities such as
\[
\int_0^\infty \rho^\beta e^{-2\rho} d\rho, \tag{54}
\]
where \(\beta\) is an integer. With the change of variable \(2\rho = t\) the integral in Eq. (54) is a gamma function
\[
\int_0^\infty \rho^\beta e^{-2\rho} d\rho = \frac{1}{2^{\beta+1}} \Gamma(\beta + 1). \tag{55}
\]
Equation (55) allows any normalization constant to be calculated.

VI. CONCLUDING REMARKS

We have constructed in a detailed realization of an su(1,1) algebra for the non-relativistic hydrogen atom by introducing the Hermitian operators \(\Omega_i, i = 1, 2, 3, \ldots\). This algebra gives the energy spectrum and the radial eigenfunctions of the system. To define the algebra we introduced the phase as an extra variable of the problem. The phase operator has been used for solving relativistic problems, such as the Dirac hydrogen atom in what is almost a transcription of the steps detailed here, and may be used to solve the one and the non-relativistic and relativistic three-dimensional Dirac–Morse problem. These problems may be given as a challenge to advanced students.

We are able to calculate the set of eigenfunctions for a fixed value of \(\ell\) by applying an operator with only a first order derivative, namely \(\Omega_3\), to a given eigenfunction. This procedure is relatively simple, at least for the first few states. The result is of intrinsic interest because we don’t need to solve the complicated Laguerre equation. This feature can make this approach suitable for an intermediate, or perhaps elementary, quantum mechanics course.

Note that in our approach, the angular momentum quantum number \(\ell\) plays a more conspicuous role than the principal quantum number \(N\) in the sense that we are forced to fix \(\ell\) first and only then proceed to calculate the wave functions. Thus, the rotational symmetry plays a central role in the behavior of the system. This point is not always apparent when dealing with the radial Hamiltonian of the hydrogen atom, but is present from the beginning because it contributes the centrifugal potential \(\ell(\ell + 1)/r^2\) to the radial equation. Here the method prominently exhibits this feature of the problem.

The algebraic operator solution we have described may serve to introduce students to diverse techniques of quantum theory, from the use of Lie algebras and symmetries, to the way of taking advantage of them to uncover hidden properties of a Hamiltonian, or to attempt, with the help of appropriate references, the extension of the method to other systems of interest. It is noteworthy that the su(1,1) Lie algebra is of great importance in quantum optics.

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[3] To obtain the last expression in Eq. (17) it is better to first calculate the product \(\Omega_1 \Omega_3\) and obtain the final result from the relation \(\Omega_3 = -\Omega_1 \Omega_3 + \Omega_3 \Omega_1 + 1\).
[4] A compact interval of real numbers is one that includes its two end points and is bounded.
[5] Note that if the \(\lambda < 0\) possibility is selected, it can be easily shown that \(\lambda\) is bounded above by \(\lambda_{\text{max}}\). Next, if we use \(\Omega_1 \Omega_3 (\Omega_{\text{max}}) = 0\), we are necessarily led to non-normalizable solutions of the form \(\sim e^{\lambda_{\text{max}}r^{1/2}} e^{\lambda_{\text{max}}r^{1/2}}\), solutions that are inappropriate for describing states of the hydrogen atom. See Ref. 16 for the details.
Foucault’s Disk. Metallic conductors passing rapidly through magnetic fields have eddy currents induced in them. Since the conductors have a finite resistivity, there will be ohmic heating of the conductors and their temperature will increase. This effect was studied by Leon Foucault (1819–1868). In his experiments, a geared-up copper wheel was rotated in the gap of an electromagnet. With no current through the magnet coils the wheel rotated easily. When the magnet was energized, resistance to rotation was felt and the copper disk started to heat up. In the 19th century, eddy currents were sometimes called Foucault currents. This early 20th century apparatus by Max Kohl of Chemnitz, Germany is at Washington and Jefferson College. (Photograph and Notes by Thomas B. Greenslade, Jr., Kenyon College.)